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Inequalities for the Derivative of a Polynomial

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1. INTRODUCTION AND STATEMENT OF RESULTS

The following result is due to Turán [8].

THEOREM A. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is sharp and equality holds in (1.1) if all the zeros of $p(z)$ lie on $|z|=1$.

More generally if the polynomial $p(z)$ has all its zeros in $|z| \leqslant K \leqslant 1$, it was proved by Malik [7] that the inequality (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+K} \max_{|z|=1} |p(z)|. \tag{1.2}$$

Malik [7] in fact deduces it by applying the following result (for another proof see [5, Theorem C, p. 503]) to the polynomial $z^n p(1/z)$.

THEOREM B. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geqslant 1$, then for $|z| \leqslant 1$,*

$$|p'(z)| \leqslant \frac{n}{1+K} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Equality in (1.3) holds for the polynomial $p(z) = (z+K)^n$.

The case when $p(z)$ has all its zeros in $|z| < K$, $K \geqslant 1$, was settled by Govil [4], who proved

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THEOREM C. *If $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $p(z)$ has all its zeros in the disk $|z| \leqslant K$, $K \geqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+K^n} \max_{|z|=1} |p(z)|. \tag{1.4}$$

The result is best possible with equality for the polynomial $p(z)=z^n+K^n$.

Although the above result is sharp, it still has two drawbacks, as is very easy to see. First, the bound in (1.4) depends only on the zero of largest modulus and not on other zeros even if some of them are very close to the origin. For example, for both the polynomials $p_1(z)=(z+l)^n$ and $p_2(z)=z^{n-1}(z+l)$, where l is an arbitrary positive number, Theorem C will give the same bound, $n/(1+l^n)$, although the polynomial $p_2(z)$ has $(n-1)$ zeros at the origin and only one zero of modulus l. Second, since the extremal polynomial in (1.4) is (z^n+K^n), it should be possible to obtain a sharper bound for polynomials $\sum_{v=0}^{n} a_v z^v$, where not all the coefficients a_1, $a_2, ..., a_{n-1}$ are zero. In other words it would be interesting to obtain a bound in Theorem C which depends on the location of all the zeros of the polynomial $\sum_{v=0}^{n} a_v z^v$ and also on the coefficients $a_1, a_2, ..., a_n$. In this connection, we prove the following

THEOREM. *Let $p(z)=\sum_{v=0}^{n} a_v z^v = a_n \prod_{v=1}^{n} (z-z_v)$, $a_n \neq 0$ be a polynomial of degree $n \geqslant 2$, $|z_v| \leqslant K_v$, $1 \leqslant v \leqslant n$, and let $K=\max(K_1, K_2, ..., K_n) \geqslant 1$. Then*

$$\begin{aligned}\max_{|z|=1} |p'(z)| \geqslant{}& \frac{2}{(1+K^n)} \left(\sum_{v=1}^{n} \frac{K}{K+K_v} \right) \max_{|z|=1} |p(z)| \\ &+ \frac{2\,|a_{n-1}|}{(1+K^n)} \sum_{v=1}^{n} \frac{1}{(K+K_v)} \left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2} \right) \\ &+ |a_1|\,(1-1/K^2), \qquad \text{if} \quad n>2\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\max_{|z|=1} |p'(z)| \geqslant{}& \frac{2}{1+K^n} \sum_{v=1}^{n} \frac{K}{K+K_v} \max_{|z|=1} |p(z)| \\ &+ \frac{(K-1)^n}{1+K^n} |a_1| \sum_{v=1}^{n} \frac{1}{K+K_v} \\ &+ (1-1/K)\,|a_1|, \qquad \text{if} \quad n=2.\end{aligned} \tag{1.6}$$

In (1.5) and (1.6) equality holds for $p(z)=z^n+K^n$.

The case when the polynomial $p(z)$ is of degree 1 is uninteresting because in that case trivially $\max_{|z|=1} |p'(z)| = (1/(1+K)) \max_{|z|=1} |p(z)|$, where K is the modulus of the zero of $p(z)$.

Since $K/(K+K_\nu) \geq \frac{1}{2}$ for $1 \leq \nu \leq n$, the above theorem gives in particular

COROLLARY. *If $p(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$, $a_n \neq 0$, is a polynomial of degree n having all its zeros in $|z| \leq K$, where $K \geq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{(1+K^n)} \max_{|z|=1} |p(z)| + \frac{n\,|a_{n-1}|}{(1+K^n)K}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right) + |a_1|\,(1-1/K^2), \qquad \text{if} \quad n>2; \tag{1.7}$$

and

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |p(z)| + \frac{(K-1)^n}{K(1+K^n)} |a_1| + (1-1/K)\,|a_1|, \qquad \text{if} \quad n=2. \tag{1.8}$$

In (1.7) and (1.8), equality holds for $p(z) = z^n + K^n$.

It is easy to verify that if $K>1$ and $n>2$, then $((K^n-1)/n - (K^{n-2}-1)/(n-2))>0$, hence for polynomials of degree >1, (1.7) and (1.8) together provide a refinement of Theorem C. In fact, excepting the case when $p(z)$ has all its zeros on $|z|=K$, $a_1=0$, and $a_{n-1}=0$, the bound obtained by our theorem is always sharper than the bound obtained from Theorem C. As is easy to see, our theorem also provides a refinement and generalization of the following result due to Aziz [1, Theorem 1].

THEOREM D [1]. *If all the zeros of the polynomial $p(z) = \prod_{\nu=1}^{n} (z - z_\nu)$ of degree n lie in $|z| \leq K$, where $K \geq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{2}{1+K^n} \sum_{\nu=1}^{n} \frac{K}{(K+|z_\nu|)} \max_{|z|=1} |p(z)|. \tag{1.9}$$

The result is best possible and equality in (1.9) holds for $p(z) = z^n + K^n$.

2. Lemmas

For the proof of the theorems, we need the following lemmas.

Lemma 1. *If* $p(z)=a_n\prod_{v=1}^{n}(z-z_v)$ *is a polynomial of degree n having all its zeros in* $|z|\leqslant 1$, *then*

$$\max_{|z|=1}|p'(z)|\geqslant\sum_{v=1}^{n}\frac{1}{(1+|z_v|)}\max_{|z|=1}|p(z)|. \tag{2.1}$$

There is equality in (2.1) *if the zeros are all positive.*

This result is due to Giroux, Rahman, and Schmeisser [3, Theorem 5]. The following result is well known and is due to Lax [6].

Lemma 2. *If* $p(z)=a_n\prod_{v=1}^{n}(z-z_v)$, $a_n\neq 0$, *is a polynomial of degree n*, $|z_v|\geqslant 1$ *for* $1\leqslant v\leqslant n$, *then*

$$\max_{|z|=1}|p'(z)|\leqslant\frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2.2}$$

There is equality in (2.2) *for the polynomial* $p(z)=1+z^n$.

Lemma 3. *If* $p(z)=\sum_{v=0}^{n}a_vz^v$ *is a polynomial of degree* $n\geqslant 2$, *then for all* $R>1$,

$$\max_{|z|=R}|p(z)|\leqslant R^n\max_{|z|=1}|p(z)|-(R^n-R^{n-2})\,|p(0)|. \tag{2.3}$$

The above result is due to Frappier, Rahman, and Ruscheweyh [2, Theorem 2].

Lemma 4. *If* $p(z)=a_n\prod_{v=1}^{n}(z-z_v)$, $a_n\neq 0$, *is a polynomial of degree* $n>2$, $|z_v|\geqslant 1$ *for* $1\leqslant v\leqslant n$, *then for* $R\geqslant 1$,

$$\max_{|z|=R}|p(z)|\leqslant\frac{(R^n+1)}{2}\max_{|z|=1}|p(z)|-|a_1|\left(\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\right). \tag{2.4}$$

Equality in (2.4) *holds for* $p(z)=(z^n+1)$.

Proof of Lemma 4. For each ϕ, $0\leqslant\phi<2\pi$, we have

$$p(Re^{i\phi})-p(e^{i\phi})=\int_1^R e^{i\phi}p'(re^{i\phi})\,dr,$$

which gives

$$|p(Re^{i\phi})-p(e^{i\phi})|\leqslant\int_1^R|p'(re^{i\phi})|\,dr. \tag{2.5}$$

Now applying first Lemma 3 and then Lemma 2 to the polynomial $p'(z)$, which is of degree $\geqslant 2$, we get

$$|p(Re^{i\phi})-p(e^{i\phi})| \leqslant \frac{n}{2}\left(\int_1^R r^{n-1}\,dr\right)\max_{|z|=1}|p(z)|$$

$$-\int_1^R (r^{n-1}-r^{n-3})\,dr\,|a_1|$$

$$=\frac{(R^n-1)}{2}\max_{|z|=1}|p(z)|-|a_1|\left(\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\right),$$

from which Lemma 4 follows.

3. Proof of the Theorem

First, we prove inequality (1.5). Here the polynomial $p(z)$ is of degree >2. Because the zeros of the polynomial $p(z)$ are z_ν $(1 \leqslant \nu \leqslant n)$, the zeros of the polynomial $P(z)=p(Kz)$ are z_ν/K $(1 \leqslant \nu \leqslant n)$ and because the polynomial $p(z)$ has all its zeros in $|z| \leqslant K$, the polynomial $P(z)$ has all its zeros in $|z| \leqslant 1$, and therefore by Lemma 1,

$$\max_{|z|=1}|P'(z)| \geqslant \sum_{\nu=1}^{n}\frac{1}{(1+|z_\nu|/K)}\max_{|z|=1}|P(z)|, \tag{3.1}$$

which is clearly equivalent to

$$K\max_{|z|=K}|p'(z)| \geqslant \sum_{\nu=1}^{n}\frac{K}{(K+|z_\nu|)}\max_{|z|=K}|p(z)|. \tag{3.2}$$

Since the polynomial $p(z)$ is of degree >2, the polynomial $p'(z)$ is of degree $\geqslant 2$; hence, applying Lemma 3 to $p'(z)$, we get, for $K \geqslant 1$,

$$\max_{|z|=K}|p'(z)| \leqslant K^{n-1}\max_{|z|=1}|p'(z)|-(K^{n-1}-K^{n-3})\,|a_1|. \tag{3.3}$$

Equation (3.3) when combined with Eq. (3.2) gives, for $K \geqslant 1$,

$$K^n\max_{|z|=1}|p'(z)|-(K^n-K^{n-2})\,|a_1| \geqslant \sum_{\nu=1}^{n}\frac{K}{(K+|z_\nu|)}\max_{|z|=K}|p(z)|. \tag{3.4}$$

Let $q(z)=z^n p(1/z)$ be the reciprocal polynomial of the polynomial $p(z)$. Since the polynomial $p(z)$ has all its zeros in $|z| \leqslant K$, $K \geqslant 1$, the polynomial

$q(z/K)$ has all its zeros in $|z| \geqslant 1$; hence, applying Lemma 4 to the polynomial $q(z/K)$, we get, for $K \geqslant 1$,

$$\max_{|z|=K} |q(z/K)| \leqslant \frac{(K^n+1)}{2} \max_{|z|=1} |q(z/k)| - \frac{|a_{n-1}|}{K}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right),$$

which is equivalent to

$$\max_{|z|=K} |p(z)| \geqslant \frac{2K^n}{(1+K^n)} \max_{|z|=1} |p(z)| + \frac{2\,|a_{n-1}|\,K^{n-1}}{1+K^n}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right). \tag{3.5}$$

On combining (3.5) with (3.4), we get

$$\frac{K^n}{\sum_{v=1}^n \frac{K}{(K+|z_v|)}} \max_{|z|=1} |p'(z)| - \frac{(K^n-K^{n-2})}{\sum_{v=1}^n \frac{K}{(K+|z_v|)}} |a_1|$$
$$\geqslant \frac{2K^n}{(1+K^n)} \max_{|z|=1} |p(z)| + \frac{2\,|a_{n-1}|\,K^{n-1}}{1+K^n}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right),$$

which gives

$$\max_{|z|=1} |p'(z)| \geqslant \frac{2}{1+K^n} \sum_{v=1}^n \frac{K}{K+|z_v|} \max_{|z|=1} |p(z)| + \frac{2\,|a_{n-1}|}{(1+K^n)}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right) \sum_{v=1}^n \frac{1}{K+|z_v|} + |a_1|\,(1-1/K^2)$$
$$\geqslant \frac{2}{(1+K^n)} \sum_{v=1}^n \frac{K}{(K+K_v)} \max_{|z|=1} |p(z)| + \frac{2\,|a_{n-1}|}{(1+K^n)} \sum_{v=1}^n \frac{1}{(K+K_v)}\left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2}\right) + |a_1|\,(1-1/K^2), \tag{3.6}$$

which is (1.5).

The proof of (1.6) follows on the same lines as the proof of (1.5) but instead of Lemma 3 it uses the inequality

$$M(p, R) \leqslant RM(p, 1) - (R-1)\,|p(0)|, \tag{3.7}$$

true for polynomials of degree 1, and instead of Lemma 4, the result corresponding to Lemma 4 for polynomials of degree 2. We omit the details.

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REFERENCES

1. ABDUL AZIZ, Inequalities for the derivative of a polynomial, *Proc. Amer. Math. Soc.* **89** (1983), 259–266.
2. C. FRAPPIER, Q. I. RAHMAN, AND ST. RUSCHEWEYH, New inequalities for polynomials, *Trans. Amer. Math. Soc.* **288** (1985), 69–99.
3. A. GIROUX, Q. I. RAHMAN, AND G. SCHMEISSER, On Bernstein's inequality, *Canad. J. Math.* **31** (1979), 347–353.
4. N. K. GOVIL, On the derivative of a polynomial, *Proc. Amer. Math. Soc.* **41** (1973), 543–546.
5. N. K. GOVIL AND Q. I. RAHMAN, Functions of exponential type not vanishing in a half-plane and related polynomials, *Trans. Amer. Math. Soc.* **137** (1969), 501–517.
6. P. D. LAX, Proof of a conjecture of P. Erdős on the derivative of a polynomial, *Bull. Amer. Math. Soc.* **50** (1944), 509–513.
7. M. A. MALIK, On the derivative of a polynomial, *J. London Math. Soc.* (2) (1969), 57–60.
8. P. TURÁN, Uber die Ableitung von Polynomen, *Compositio Math.* **7** (1939), 89–95.